Examples of spaces with branching geodesics satisfying the curvature-dimension condition

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July 17, 2013

Abstract

In this short note, we give two simple examples of metric measure spaces containing branching geodesics and satisfying Lott, Sturm and Villani's curvature-dimension condition. In the first example, only geodesics in a subspace of co-dimension 1 can branch. The second one is positively curved in the sense of the curvature-dimension condition. We also discuss related open problems.

1 Background

The aim of this half-expository note is to draw attention to examples of spaces satisfying the curvature-dimension condition and containing branching geodesics. We first explain the background of this subject.

1.1 Curvature-dimension condition

The *curvature-dimension condition* for a (complete, separable) metric measure space (X, d, m) is a generalized notion of lower Ricci curvature bounds introduced independently by Sturm ([St1], [St2]) and Lott and Villani ([LV1], [LV2]). Given $K \in \mathbb{R}$ and $N \in (1, \infty]$, a Riemannian manifold (M, g) equipped with the Riemannian volume measure satisfies the curvature-dimension condition $\mathsf{CD}(K, N)$ if and only if $\mathrm{Ric}_g \geq K$ (i.e., $\mathrm{Ric}_g(v, v) \geq K\langle v, v\rangle$ for all $v \in TM$) and $\dim M \leq N$. Metric measure spaces satisfying $\mathsf{CD}(K, N)$ are known to behave like enjoying 'Ric $\geq K$ and dim $\leq N$' in geometric and analytic respects (see [St1], [St2], [LV1], [LV2] and [Vi, Part III]).

The curvature-dimension condition is a certain convexity condition of an entropy functional on the space of probability measures on X. Let us give the precise definition (in the sense of Sturm [St1]) only for $N = \infty$. Denote by $\mathcal{P}(X)$ the set of all Borel probability measures on (X, d), and by $\mathcal{P}^2(X)$ the subset consisting of $\mu \in \mathcal{P}(X)$ with finite second moment. Define the L^2-*Wasserstein distance* between $\mu, \nu \in \mathcal{P}^2(X)$ by

$$W_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{X \times X} d(x, y)^2 \, \pi(dxdy) \right)^{1/2},$$

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Supported in part by the Grant-in-Aid for Young Scientists (B) 23740048.

where $\Pi(\mu, \nu)$ is the set of all $\pi \in \mathcal{P}(X \times X)$ whose first and second projections to X coincide with μ and ν, respectively. We call $(\mathcal{P}^2(X), W_2)$ the L^2-*Wasserstein space* over (X, d). The *relative entropy* $\mathrm{Ent}_m : \mathcal{P}^2(X) \longrightarrow [-\infty, \infty]$ associated with a Borel measure m on X is defined by

$$\mathrm{Ent}_m(\mu) := \int_X \rho \log \rho \, dm$$

if $\mu = \rho m \ll m$ and $\int_{\{\rho > 1\}} \rho \log \rho \, dm < \infty$, otherwise we set $\mathrm{Ent}_m(\mu) := \infty$. Then (X, d, m) is said to satisfy $\mathsf{CD}(K, \infty)$ if any pair $\mu, \nu \in \mathcal{P}^2(X)$ is connected by a *minimal geodesic* $(\mu_t)_{t \in [0,1]} \subset \mathcal{P}^2(X)$ (i.e., $\mu_0 = \mu$, $\mu_1 = \nu$, $W_2(\mu_s, \mu_t) = |s - t| W_2(\mu, \nu)$ for all $s, t \in [0, 1]$) satisfying

$$\mathrm{Ent}_m(\mu_t) \le (1 - t)\, \mathrm{Ent}_m(\mu) + t\, \mathrm{Ent}_m(\nu) - \frac{K}{2}(1 - t) t W_2(\mu, \nu)^2 \tag{1.1}$$

for all $t \in [0, 1]$. A minimal geodesic $(\mu_t)_{t \in [0,1]}$ in $(\mathcal{P}^2(X), W_2)$ can be described via the push-forward of μ_0 along geodesics in X, so that the behavior of $\mathrm{Ent}_m(\mu_t)$ is naturally controlled by the Ricci curvature in the Riemannian case. The definition of $\mathsf{CD}(K, N)$ for $N < \infty$ is more involved (especially for $K \neq 0$), here we only mention that $\mathsf{CD}(K, N)$ implies $\mathsf{CD}(K', N')$ for any $K' \in (-\infty, K]$ and $N' \in [N, \infty]$.

1.2 Branching geodesics

The equivalence between the curvature-dimension condition and lower Ricci curvature bounds was extended to Finsler manifolds in [Oh3, Theorem 1.2]. Precisely, given an n-dimensional Finsler manifold (M, F) with a positive $\mathcal{C}^\infty$-measure m, we consider the *weighted Ricci curvature* Ric_N for $N \in [n, \infty]$ given by

$$\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + (\psi \circ \eta)''(0) - \frac{(\psi \circ \eta)'(0)^2}{N - n} \quad \text{for } v \in T_x M, \tag{1.2}$$

where Ric is the Finsler-Ricci curvature, η is the geodesic with $\dot{\eta}(0) = v$, and ψ is determined by $m = e^{-\psi} \mathrm{vol}_{\dot{\eta}}$ around x with the volume measure $\mathrm{vol}_{\dot{\eta}}$ of the Riemannian metric $g_{\dot{\eta}}$ induced from $\dot{\eta}$ (see [Oh3] for the precise definition). Then (M, F, m) satisfies $\mathsf{CD}(K, N)$ if and only if $\mathrm{Ric}_N(v) \ge K F(v)^2$ for all $v \in TM$. By combining this equivalence with the stability of $\mathsf{CD}(K, N)$ under convergence of metric measure spaces, all normed spaces $(\mathbb{R}^n, |\cdot|)$ with the Lebesgue measure $m_{\mathbf{L}}$ satisfy $\mathsf{CD}(0, n)$ (see also [Vi, Theorem in page 908]). In particular, the ℓ_∞-space $(\mathbb{R}^n, |\cdot|_\infty, m_{\mathbf{L}})$ satisfies $\mathsf{CD}(0, n)$.

Geodesics in $(\mathbb{R}^n, |\cdot|_\infty)$ can *branch* in the sense that there exist two minimal geodesics $\eta_1, \eta_2 : [0, 1] \longrightarrow \mathbb{R}^n$ with respect to $|\cdot|_\infty$ such that $\eta_1(t) = \eta_2(t)$ on $[0, \varepsilon]$ for some $\varepsilon \in (0, 1)$ and that $\eta_1(t) \neq \eta_2(t)$ at some $t \in (\varepsilon, 1]$. The existence of branching geodesics causes difficulties in the study of spaces satisfying the curvature-dimension condition. Although the non-branching assumption has been recently removed in some cases (see [Ra1], [Ra2]), there still remain important results for those we need to assume that geodesics do not branch. See, for instances, [St1, Proposition 4.16] and [DS] for products of CD-spaces, and [Oh2, §5] for a maximal diameter rigidity. If geodesics do not branch (then (X, d) is called a *non-branching space*), then there is essentially only one minimal geodesic between absolutely continuous measures $\mu, \nu \ll m$ (see [St2, Lemma 4.1]), so that the entropy is convex along *every* minimal geodesic. This enables us to localize $\mathsf{CD}(K,N)$ to a certain inequality (concavity of Jacobian) along almost all geodesics in X describing the minimal geodesic between μ and ν (see [St2, Proposition 4.2]). Such an infinitesimal inequality is sometimes necessary for sharper estimates.

As far as the author knows, only known examples of spaces satisfying $\mathsf{CD}(K,N)$ and containing branching geodesics are normed spaces. We shall present two more examples of branching spaces (built as *singular* Finsler manifolds) in this note, whereas they are still modifications of normed spaces in a sense. We also discuss open problems related to these examples.

2 An essentially non-branching example

2.1 Construction

Given a bounded convex domain $D \subset \mathbb{R}^n$, Hilbert [Hi] introduced the distance function

$$d_{\mathcal{H}}(x,y) := \frac{1}{2} \log\left(\frac{|x'-y| \cdot |x-y'|}{|x'-x| \cdot |y-y'|} \right) \quad \text{for } x,y \in D,\ x \neq y,$$

where $|\cdot|$ is the Euclidean norm, and $x' = x + s(y-x)$ and $y' = x + t(y-x)$ are the intersections of the boundary ∂D and the line passing through x and y with $s < 0 < t$ (see Figure 1). This is a generalization of the Klein model of a hyperbolic space which corresponds to the case where D is the unit ball. It is easily seen that, for any $x, y \in D$, the line segment $\eta(t) := x + t(y-x)$ ($t \in [0,1]$) gives a minimal geodesic from x to y with respect to $d_{\mathcal{H}}$. If D is strictly convex (i.e., $x, y \in \partial D$ implies $(x+y)/2 \in D$ unless $x = y$), then η is the unique minimal geodesic between x and y. Otherwise, minimal geodesics may not be unique. For instance, consider the square $D = (-1,1)^2 \subset \mathbb{R}^2$. Since $d_{\mathcal{H}}((0,0),(a,b)) = d_{\mathcal{H}}((0,0),(a,b'))$ for any $a \in (0,1)$ and $b, b' \in [-a,a]$, we can construct uncountably many minimal geodesics between $(0,0)$ and $(a,0)$ with $a > 0$ (e.g., thick broken segments in Figure 2).

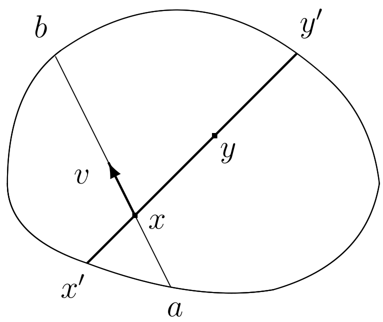

Figure 1

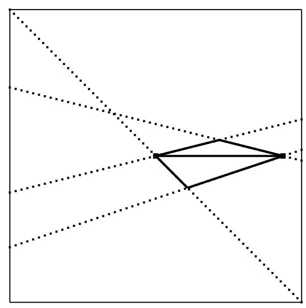

Figure 2

If ∂D is $\mathcal{C}^\infty$ and positively curved (the latter will be called the *strong convexity* of D), then $d_{\mathcal{H}}$ is realized by the $\mathcal{C}^\infty$-Finsler metric

$$F_{\mathcal{H}}(x,v) := \frac{|v|}{2}\left(\frac{1}{|x-a|}+\frac{1}{|x-b|}\right) \quad \text{for } v \in T_xD = \mathbb{R}^n,$$

where $a = x+sv$ and $b = x+tv$ are the intersections of ∂D and the line passing through x in the direction v with $s < 0 < t$ (see Figure 1). It is well-known that the Finsler manifold $(D, F_{\mathcal{H}})$ has the constant flag curvature -1 (see [Sh, §12.2]). Recently, the author showed that $(D, F_{\mathcal{H}})$ equipped with the Lebesgue measure $m_{\mathbf{L}}$ on D has the bounded weighted Ricci curvature:

$$\mathrm{Ric}_\infty(v) \in \bigl(-(n-1), 2\bigr], \qquad \mathrm{Ric}_N(v) \in \left(-(n-1)-\frac{(n+1)^2}{N-n}, 2\right]$$

for any unit vector $v \in TD$ and $N \in (n,\infty)$ ([Oh4, Theorem 1.2]). Thus, for any bounded convex domain $D \subset \mathbb{R}^n$, $(D, d_{\mathcal{H}}, m_{\mathbf{L}})$ satisfies $\mathsf{CD}(K,N)$ with

$$K = -(n-1) \quad \text{for } N = \infty, \qquad K = -(n-1) - \frac{(n+1)^2}{N-n} \quad \text{for } N \in (n,\infty)$$

by approximating D by strongly convex domains with $\mathcal{C}^\infty$-boundaries.

Now, let us consider a domain $D \subset \mathbb{R}^3$ such that $P := D \cap (\mathbb{R}^2 \times \{0\}) = (-1,1)^2 \times \{0\}$ and that $\partial D \setminus (\mathbb{R}^2 \times \{0\})$ is $\mathcal{C}^\infty$ and positively curved. Then P contains branching geodesics, while geodesics not included in P do not branch. Indeed, for any $x, y \in D$ with $x \notin P$ or $y \notin P$, the line segment is a unique minimal geodesic between them. Hence $(D, d_{\mathcal{H}}, m_{\mathbf{L}})$ is essentially non-branching in the sense of Theorem 2.2 below. A concrete example of such a domain is

$$D_\square := \{(a,b,c) \in \mathbb{R}^3 \mid (a,b) \in (-1,1)^2,\ c^2 < (1-a^2)(1-b^2)\}. \tag{2.1}$$

Remark 2.1 In [Oh4], we also gave a lower bound of the weighted Ricci curvature for Funk's distance [Fu]:

$$d_{\mathcal{F}}(x,y) = \log\left(\frac{|x-y'|}{|y-y'|}\right) \quad \text{for } x, y \in D,\ x \neq y,$$

which can be regarded as a 'non-symmetrization' of $d_{\mathcal{H}}$. In $(D_\square, d_{\mathcal{F}})$, however, geodesics in a plane containing only one edge of ∂P (e.g., $\{1\} \times [-1,1] \times \{0\}$) can branch. Therefore there are many more branching geodesics than $(D_\square, d_{\mathcal{H}})$.

2.2 Open problems

The above example is inspired by the following recent work of Rajala and Sturm. We say that (X,d,m) satisfies the *strong curvature-dimension condition* $\mathsf{sCD}(K,\infty)$ if (1.1) holds along any minimal geodesic in $\mathcal{P}^2(X)$.

Theorem 2.2 ([RS, Corollary 1.2]) *Let (X,d) be a complete separable metric space with a locally finite measure m on X, and suppose that (X,d,m) satisfies $\mathsf{sCD}(K,\infty)$. Then any minimal geodesic between absolutely continuous measures $\mu_0, \mu_1 \in \mathcal{P}^2(X)$ is concentrated on a set of non-branching geodesics.*

See [RS] for the precise statement. Our example $(D_\square, d_{\mathcal{H}}, m_{\mathbf{L}})$ in (2.1) satisfies the hypothesis of Theorem 2.2, thus it shows that geodesics still can branch in this kind of *essentially non-branching* spaces.

An important class of spaces covered by Theorem 2.2 is the one of metric measure spaces enjoying the *Riemannian curvature-dimension condition* $\mathsf{RCD}(K,\infty)$ introduced by Ambrosio, Gigli and Savaré [AGS]. Roughly speaking, $\mathsf{RCD}(K,\infty)$ is defined by the combination of $\mathsf{sCD}(K,\infty)$ and the linearity of the *heat flow*. The heat flow on a Finsler manifold (M,F,m) is linear if and only if F comes from a Riemannian metric (see [OS]), so that $(D_\square, d_{\mathcal{H}}, m_{\mathbf{L}})$ does not satisfy $\mathsf{RCD}(K,\infty)$ for any K.

Problem A Is there a metric measure space satisfying $\mathsf{RCD}(K,\infty)$ for some $K \in \mathbb{R}$ and containing branching geodesics?

The author does not have any idea how to construct such a space. If the answer is NO, then we can remove the non-branching assumption from a number of theorems on RCD-spaces. Furthermore, since all limit spaces of Riemannian manifolds with $\mathrm{Ric} \ge K$ satisfy $\mathsf{RCD}(K,\infty)$, the negative answer to Problem A contributes to the development of the study of the local structure of these limit spaces. After seminal work of Cheeger and Colding [CC], it is the important open question whether such limit spaces admit branching geodesics or not.

Problem B In the case where the answer to Problem A is YES, can one construct a branching space satisfying $\mathsf{RCD}(K,\infty)$ as a limit space of Riemannian manifolds with $\mathrm{Ric} \ge K$?

This is related to another deep question: Can any metric measure space satisfying $\mathsf{RCD}(K,\infty)$ be realized as the limit of a sequence of Riemannian manifolds with $\mathrm{Ric} \ge K$ (or weighted Riemannian manifolds with $\mathrm{Ric}_\infty \ge K$)?

3 A positively curved example

3.1 Construction

We next construct a branching space satisfying $\mathsf{CD}(K,N)$ with $K>0$ and $N<\infty$. Fix $N>2$, $R>0$ and $a \in (0,(N-2)R^{-2})$. Consider the metric measure space (X_R, d_∞, m_a) given by

$$X_R := \{x \in \mathbb{R}^2 \,|\, |x|_2 < R\}, \quad d_\infty(x,y) := |x-y|_\infty, \quad m_a := \exp\left(-\frac{a|x|_2^2}{2}\right) m_{\mathbf{L}}(dx).$$

Clearly geodesics can branch in (X_R, d_∞). Note that, along any line segment η in X_R,

$$\left(\frac{a|\eta|_2^2}{2}\right)' \le a|\eta|_2 \cdot |\dot\eta|_2 \le aR|\dot\eta|_2, \qquad \left(\frac{a|\eta|_2^2}{2}\right)'' = a|\dot\eta|_2^2.$$

Since $|\dot\eta|_2 \ge |\dot\eta|_\infty$, by approximating $|\cdot|_\infty$ by $\mathcal{C}^\infty$ and strongly convex norms to be precise, we see that (X_R, d_∞, m_a) satisfies $\mathsf{CD}(K,N)$ with (recall (1.2))

$$K = a - \frac{(aR)^2}{N-2}.$$

Note that $K > 0$ by the choice of a. We similarly find that $(\mathbb{R}^2, d_\infty, m_a)$ satisfies $\mathsf{CD}(a, \infty)$ for any $a > 0$.

3.2 Open problems

The above example is related to a rigidity result for positively curved spaces. By the Bonnet–Myers type theorem, the diameter of a metric measure space satisfying $\mathsf{CD}(K, N)$ with $K > 0$ and $N < \infty$ is at most $\pi\sqrt{(N-1)/K}$. In [Oh2, §5], under the weaker assumption of the *measure contraction property* $\mathsf{MCP}(K, N)$ (see [Oh1], [St2, §5]), we showed that a non-branching space with the maximal diameter $\pi\sqrt{(N-1)/K}$ is homeomorphic to the spherical suspension of a topological measure space. The non-branching property was necessary for proving the continuity of the homeomorphism.

In our example (X_R, d_∞), the diameter is $2R$ and is smaller than the maximal one:

$$\pi\sqrt{\frac{N-1}{K}} = \pi\sqrt{\frac{N-1}{a-(aR)^2/(N-2)}} \ge \pi\sqrt{(N-1)\frac{4R^2}{N-2}} > 2\pi R.$$

Problem C Is there a metric measure space satisfying $\mathsf{CD}(K, N)$ with $K > 0$ and $N < \infty$, attaining the maximal diameter $\pi\sqrt{(N-1)/K}$, and containing branching geodesics? If YES, can such a space be realized as the spherical suspension of some space?

One can weaken the condition $\mathsf{CD}(K, N)$ to $\mathsf{MCP}(K, N)$, or strengthen it to the combination of $\mathsf{CD}(K, N)$ and $\mathsf{RCD}(K, \infty)$. Related to the latter, we can also ask the following.

Problem D Is there a metric measure space which satisfies $\mathsf{CD}(K, N)$ with $K > 0$ (or $K = 0$), contains branching geodesics, but is essentially non-branching in the sense of Theorem 2.2?

This would be a more accessible one than other problems. Recall that the space $(D_\square, d_\mathcal{H}, m_\mathbf{L})$ in the previous section admits only a negative curvature bound.

Acknowledgements. I would like to thank Professors Karl-Theodor Sturm and Tapio Rajala for discussions during a workshop at Scuola Normale Superiore, Pisa, in November 2012. I am also grateful to the organizers of the workshop, Professors Luigi Ambrosio, Giuseppe Buttazzo and Giuseppe Savaré, for their hospitality.

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